Asymptotic Solution Near Focus of a Two-dimensional Shock and Instability Near Axis of a Cylindrical Shock with CCW Approach

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The Chester-Chisnell-Whitham equations in Cartesian and polar coordinates are written out for the two-dimensional case. The solution expanded near a point is obtained. The asymptotic solutions of order 2 near focus are presented. Linear small perturbation analysis for a converging cylindrical shock is given. To study the development of perturbations numerical solutions are carried out.

1. The Shock Dynamics and CCW Approach

Let us consider the propagation of two-dimensional plane shock waves into a static and uniform medium. Whitham [1, 2] has utilized orthogonal shock-ray coordinates where the shock positions are determined by the curves $\alpha = \text{const}$, and the rays by $\beta = \text{const}$, taking

$$\alpha = a_0 t, \tag{1}$$

where a_0 denotes the sound speed in the undisturbed medium and t the time. The shock Mach number M represents the Lamé coefficient in α-direction and L the other one in β-direction, which represents the width of the differential ray channel. From the orthogonality of α and β follow the relations

$$\frac{\partial \theta}{\partial \beta} = \frac{1}{M} \frac{\partial L}{\partial \alpha}, \tag{2}$$

$$\frac{\partial \theta}{\partial \alpha} = -\frac{1}{L} \frac{\partial M}{\partial \beta}, \tag{3}$$

where θ is the angle made by the ray with a fixed direction. For unit height of the cylindrical shock the cross-sectional area of the differential ray channel A is equal to L. Eliminating θ, we have

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{M} \frac{\partial A}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{A} \frac{\partial M}{\partial \beta}\right) = 0. \tag{4}$$

Originally the shock propagation is closely coupled with the flow field behind the shock. Whitham suggested a dynamic approximation: if an approximate functional relation between A and M is given instead of the coupling, then the motion of the shock can be solved independently from the flow field behind the shock. This summarizes the essence of Whitham's shock dynamics.

For channels Chester [3] and Chisnell [4] obtained the following approximate relation:

$$\frac{\mathrm{d}A}{A} = -\frac{2M\,\mathrm{d}M}{(M^2-1)\,K(M)}, \tag{5}$$

where (6)

$$K(M) = 2\left[\left(1 + \frac{2}{\gamma+1}\,\frac{1-\mu^2}{\mu}\right)(2\mu + 1 + M^{-2})\right]^{-1},$$

$$\mu^2 = \frac{(\gamma-1)\,M^2+2}{2\gamma M^2 - (\gamma-1)} \tag{7}$$

and γ stands for the specific heat ratio.

Whitham applied this relation to the ray channels, i.e. he took the Chester-Chisnell relation as the dynamic assumption to solve the motion of the shock. This is usually called the CCW approach.

2. Shock Dynamics Equations in Cartesian Coordinates

Obviously, the following relation holds:

$$M^2 (\nabla \alpha)^2 = 1. \tag{8}$$

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Besides, a dynamic assumption can be represented also by the decay index defined by

$$G(M) = -\frac{A}{M}\frac{dM}{dA}, \tag{9}$$

thus

$$\nabla M \cdot \nabla \alpha = \frac{1}{M^2}\frac{\partial M}{\partial \alpha} = G\left(-\frac{1}{AM}\frac{\partial A}{\partial \alpha}\right). \tag{10}$$

For the two-dimensional plane case the term within parantheses represents just the curvature of the shock

$$-\frac{1}{AM}\frac{\partial A}{\partial \alpha} = -\frac{1}{L}\left(\frac{1}{M}\frac{\partial L}{\partial \alpha}\right) = -\frac{1}{L}\frac{\partial \theta}{\partial \beta} = \frac{1}{R_c}, \tag{11}$$

where R_c is the radius of curvature of the shock and the moving directions are taken as positive. Thus (10) may be written as

$$\nabla M \cdot \nabla \alpha = G/R_c, \tag{12}$$

which represents the acceleration of the shock elements, that is

$$\frac{\partial M}{\partial \alpha} = \frac{M^2 G}{R_c}. \tag{13}$$

Instead of the general Cartesian coordinates x and y we take now y and α as independent variables describing the motion of the shock by $x(y, \alpha)$ and $M(y, \alpha)$. Substituting the transformation relations

$$\begin{aligned}
&\left(\frac{\partial \alpha}{\partial x}\right)_y = 1\Big/\left(\frac{\partial x}{\partial \alpha}\right)_y, \\
&\left(\frac{\partial \alpha}{\alpha y}\right)_x = -\left(\frac{\partial x}{\partial y}\right)_\alpha \left(\frac{\partial x}{\partial \alpha}\right)_y, \\
&\left(\frac{\partial M}{\partial x}\right)_y = \left(\frac{\partial M}{\partial \alpha}\right)_y\Big/\left(\frac{\partial x}{\partial \alpha}\right)_y, \\
&\left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial M}{\partial y}\right)_\alpha - \left(\frac{\partial M}{\partial \alpha}\right)_y\left(\frac{\partial x}{\partial y}\right)_\alpha\Big/\left(\frac{\partial x}{\partial \alpha}\right)_y
\end{aligned} \tag{14}$$

and the expression

$$\begin{aligned}
1/R_c &= \pm\frac{\partial^2 x}{\partial y^2}\Big/\left[1 + \left(\frac{\partial x}{\partial y}\right)^2\right]^{3/2} \\
&= M^3\frac{\partial^2 x}{\partial y^2}\Big/\left(\left(\frac{\partial x}{\partial \alpha}\right)^3\right)
\end{aligned} \tag{15}$$

into (8) and (12), we obtain the shock dynamics equations for the plane case in Cartesian coordinates as follows:

$$\left(\frac{\partial x}{\partial \alpha}\right)^2 = M^2 + M^2\left(\frac{\partial x}{\partial y}\right)^2, \tag{16}$$

$$\begin{aligned}
\left(\frac{\partial x}{\partial \alpha}\right)^3\frac{\partial M}{\partial \alpha} = M^2\frac{\partial x}{\partial y}\frac{\partial M}{\partial y}\left(\frac{\partial x}{\partial \alpha}\right)^2 \\
+ M^5 G\frac{\partial^2 x}{\partial y^2}.
\end{aligned} \tag{17}$$

In case of plane shocks with $\partial x/\partial y = 0$, the solution of (16) and (17) is given by

$$\frac{dx}{d\alpha} = M = \text{const}.$$

From (16) and (17) and the last condition the following linear relations for small perturbations are found:

$$M^* = \frac{\partial x^*}{\partial \alpha},$$

$$\frac{\partial^2 x^*}{\partial \alpha^2} = M^2 G\frac{\partial^2 x^*}{\partial y^2},$$

where the asterisk represents the small perturbations. The last equation yields also $\sqrt{M^2 G}$, the propagation velocity of the perturbations along the shock.

3. Solution Expanded Near a Point

With the equations obtained above we discuss the character of the motion of the shock near a time-space point. If we take this point as origin and expand x and M near the point, the solution may be written as

$$\begin{aligned}
x(y, \alpha) &= \sum a_{ij}\alpha^i y^j, \\
M(y, \alpha) &= \sum b_{ij}\alpha^i y^j.
\end{aligned} \tag{18}$$

Substituting these expansions into (16) and (17) and comparing the same terms we may obtain a system of relations between a_{ij} and b_{ij}. If we take a_{0j} and b_{0j} as independent parameters, then all the other coefficients may be expressed by these independent parameters.

As a result of the selection of the origin, $a_{00} = 0$ evidently. The expansions (18) may be written as follows:

$$\begin{pmatrix} x \\ M \end{pmatrix} = \underbrace{\begin{pmatrix} a_{01}\, y + a_{10}\, \alpha \\ b_{00} \end{pmatrix}}_{\text{order } 0} + \underbrace{\begin{pmatrix} \sum\limits_{i+j=2} a_{ij}\, \alpha^i y^j \\ \sum\limits_{i+j=1} b_{ij}\, \alpha^i y^j \end{pmatrix}}_{\text{order } 1} + \underbrace{\begin{pmatrix} \sum\limits_{i+j=3} a_{ij}\, \alpha^i y^j \\ \sum\limits_{i+j=2} b_{ij}\, \alpha^i y^j \end{pmatrix}}_{\text{order } 2} + \dots .$$

The independent parameters in the terms of order 0 are b_{00} and a_{01}, but a_{01} is determined only by choosing the direction of the coordinates. If the direction of motion of the shock at the point coincides with the x-axis, then $a_{01} = 0$. Therefore there is only one essential parameter b_{00}, i.e. the shock Mach number M_0 at the point considered. The other coefficient a_{10} is determined from the $\alpha^0 y^0$-term of (16).

The independent parameters in the terms of order 1 are b_{01} and a_{02}, and the other coefficients b_{10}, a_{11} and a_{20} are determined from the $\alpha^0 y^0$-term of (17), the $\alpha^0 y^1$- and $\alpha^1 y^0$-terms of (16).

The independent parameters in the terms of order 2 are b_{02} and a_{03}, and the other coefficients b_{11}, b_{20}, a_{12}, a_{21} and a_{30} are determined from the $\alpha^0 y^1$-, $\alpha^1 y^0$-terms of (17) and the $\alpha^0 y^2$-, $\alpha\, y$-, $\alpha^2 y^0$-terms of (16).

In general, there are $2n + 3$ coefficients b_{ij} $(i + j = n)$ and a_{ij} $(i + j = n + 1)$ in the terms of order n. Taking b_{0n} and $a_{0\,n+1}$ as independent parameters, the other $2n + 1$ coefficients b_{ij} $(i > 0,\ i + j = n)$ are determined from the $\alpha^{i-1} y^j$-terms of (17) and a_{ij} $(i > 0,\ i + j = n + 1)$ from the $\alpha^{i-1} y^j$-terms of (16).

The function $G(M)$ in (17) is expanded as follows:

$$\begin{aligned} G(M) &= G_0 + G_0' (M - M_0) + \dots \\ &= G_0 + G_0' \sum_{i+j>0} b_{ij}\, \alpha^i y^j + \dots , \end{aligned}$$

where

$$G_0 = G(M_0)\,, \quad G_0' = G'(M_0), \dots .$$

The following results are obtained:

$$\text{order } 0 \quad b_{00} = M_0\,, \quad a_{01} = 0\,, \quad a_{10} = M_0\,; \tag{19}$$

$$\begin{aligned} \text{order } 1 \quad & b_{10} = 2 M_0^2\, G_0\, a_{02}\,, \\ & a_{11} = b_{01}\,, \quad a_{20} = M_0^2\, G_0\, a_{02}\,; \end{aligned} \tag{20}$$

$$\begin{aligned} \text{order } 2 \quad & b_{11} = 6 M_0^2\, G_0\, a_{03} \\ & \qquad + 2\,(1 + 2 G_0 + M_0\, G_0')\, M_0\, b_{01}\, a_{02}\,, \\ & b_{20} = M_0^2\, G_0\, b_{02} + 2\,(1 + 2 G_0 + M_0\, G_0') \\ & \qquad \cdot M_0^3\, G_0\, a_{02}^2 + \tfrac{1}{2} M_0\, b_{01}^2\,, \\ & a_{12} = b_{02} + 2 M_0\, a_{02}^2\,, \\ & a_{21} = 3 M_0^2\, G_0\, a_{03} \\ & \qquad + (2 + 2 G_0 + M_0\, G_0')\, M_0\, b_{01}\, a_{02}\,, \\ & a_{30} = \tfrac{1}{3} M_0^2\, G_0\, b_{02} + \tfrac{2}{3}\,(1 + 2 G_0 + M_0\, G_0') \\ & \qquad \cdot M_0^3\, G_0\, a_{02}^2 + \tfrac{1}{3} M_0\, b_{01}^2\,. \end{aligned} \tag{21}$$

Here, the coefficients are written out up to the terms of order 2.

4. The Asymptotic Solution of Order 2 Near Focus

Near focus the solution is distinguished by the fact that the two parameters of order 1 are equal to zero, $b_{01} = a_{02} = 0$, and then from (20) the other coefficients of order 1 are also equal to zero, $b_{10} = a_{11} = a_{20} = 0$.

If we denote the parameters of order 2 by $-k$ and e instead of b_{02} and a_{03}, the asymptotic solution of order 2 near focus is given by

$$\begin{aligned} x &= M_0\, \alpha - \tfrac{1}{3} M_0^2\, G_0\, k\, \alpha^3 - k\, \alpha\, y^2 \\ &\quad + e\, y^3 + 3 M_0^2\, G_0\, e\, \alpha^2 y\,, \\ M &= M_0 - M_0^2\, G_0\, k\, \alpha^2 - k\, y^2 + 6 M_0^2\, G_0\, e\, \alpha\, y\,. \end{aligned} \tag{22}$$

The parameter e represents the unsymmetry; we discuss generally the symmetric case, $e = 0$, and obtain the following asymptotic solution of order 2 for symmetric focusing:

$$\begin{aligned} x &= M_0\, \alpha - \tfrac{1}{3} M_0^2\, G_0\, k\, \alpha^3 - k\, \alpha\, y^2\,, \\ M &= M_0 - M_0^2\, G_0\, k\, \alpha^2 - k\, y^2\,. \end{aligned} \tag{23}$$

5. Two-dimensional Axially Symmetric Case

For the two-dimensional axially symmetric case, the axis of symmetry is taken as z-axis, the distance from the axis as r-coordinate and the relation between A and L becomes

$$A = 2\pi\, r\, L\,.$$

Instead of (11) of the plane case the following relation holds:

$$-\frac{1}{AM}\frac{\partial A}{\partial \alpha} = -\frac{1}{LM}\frac{\partial L}{\partial \alpha} - \frac{1}{rM}\frac{\partial r}{\partial \alpha}$$
$$= \frac{1}{R_c} + \frac{\sin\delta}{r} = \frac{1}{R_c} + \frac{1}{R_a}$$

(see Figure 1).

The equation corresponding to (12) is then given by

$$\nabla M \cdot \nabla \alpha = G\left(\frac{1}{R_c} + \frac{1}{R_a}\right). \tag{24}$$

Substituting the transformation relations (14), expression (15) ($x \rightarrow z$, $y \rightarrow r$) and

$$\frac{1}{R_a} = \frac{\sin\delta}{r} = \frac{1}{r}\cos\delta\,\mathrm{tg}\,\delta = \frac{1}{r}\left(\frac{1}{\frac{1}{M}\frac{\partial z}{\partial \alpha}}\right)\left(\frac{\partial z}{\partial r}\right)$$
$$= \frac{M}{r}\frac{\partial z/\partial r}{\partial z/\partial \alpha}, \tag{25}$$

the following shock dynamics equations for the axially symmetric case are obtained

$$\left(\frac{\partial z}{\partial \alpha}\right)^2 = M^2 + M^2\left(\frac{\partial z}{\partial r}\right)^2, \tag{26}$$

$$r\left(\frac{\partial z}{\partial \alpha}\right)^3 \frac{\partial M}{\partial \alpha} = r\,M^2 \frac{\partial z}{\partial r}\frac{\partial M}{\partial r}\left(\frac{\partial z}{\partial \alpha}\right)^2 \tag{27}$$
$$+ r\,M^5\,G\frac{\partial^2 z}{\partial r^2} + M^3\,G\frac{\partial z}{\partial r}\left(\frac{\partial z}{\partial \alpha}\right)^2.$$

Expansion of z and M near a point at the z-axis yields

$$z\,(r,\alpha) = \sum c_{ij}\,\alpha^i\,r^j\,, \tag{28}$$
$$M\,(r,\alpha) = \sum d_{ij}\,\alpha^i\,r^j\,.$$

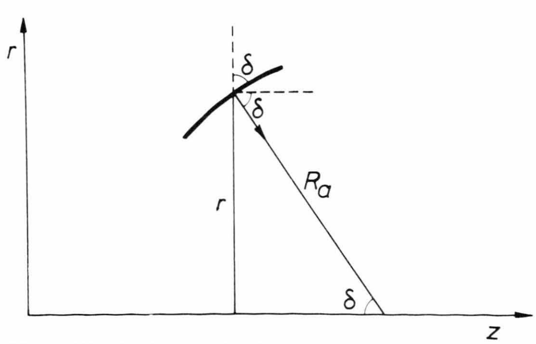


Fig. 1. Shock element in the two-dimensional axially symmetric case.

In the present case there is only one independent parameter in each order because of the axial symmetry.

The independent parameter in the terms of order 0 is $d_{00} = M_0$, and the other coefficients c_{01} and c_{10} are determined from the $\alpha^0 z^0$-terms of (26) and (27). The independent parameter in the terms of order 1 is c_{02}, and the other coefficients are determined from the $\alpha^i z^j$-terms $(i + j = 1)$ of (26) and (27). The independent parameter in the terms of order 2 is d_{02}, and the other coefficients are determined from the $\alpha^i z^j$-terms $(i + j = 2)$ of the same equations. In general, the independent parameter in the terms of order n is $c_{0\,n+1}$ if n is an odd number, or d_{0n} if n is an even number, the other $2n = 2$ coefficients are determined from the $\alpha^i z^j$-terms $(i + j = n)$ of (26) and (27).

The following results are obtained:

$$\text{order } 0 \quad d_{00} = M_0\,, \quad c_{01} = 0\,, \quad c_{10} = M_0\,; \tag{29}$$

$$\text{order } 1 \quad d_{01} = 0, \quad d_{10} = 4\,M_0^2\,G_0\,c_{02}\,, \tag{30}$$
$$c_{11} = 0\,, \quad c_{20} = 2\,M_0^2\,G_0\,c_{02}\,;$$

$$\text{order } 2 \quad d_{11} = 0\,,$$
$$d_{20} = 2\,M_0^2\,G_0\,d_{02}$$
$$+ 4\,(1 + 4\,G_0 + 2\,M_0\,G_0')\,M_0^3\,G_0\,c_{02}^2\,,$$
$$c_{03} = 0\,, \quad c_{12} = d_{02} + 2\,M_0\,c_{02}^2\,, \quad c_{21} = 0\,,$$
$$c_{30} = \tfrac{2}{3}\,M_0^2\,G_0\,d_{02} \tag{31}$$
$$+ \tfrac{4}{3}\,(1 + 4\,G_0 + 2\,M_0\,G_0')\,M_0^3\,G_0\,c_{02}^2\,.$$

Near focus there is $c_{02} = 0$, and then all coefficients of order 1 are equal to 0; thus the asymptotic solution of order 2 for axially symmetric focusing is obtained as follows:

$$z = M_0\,\alpha - \tfrac{2}{3}\,M_0^2\,G_0\,k\,\alpha^3 - k\,\alpha\,r^2\,, \tag{32}$$
$$M = M_0 - 2\,M_0^2\,G_0\,k\,\alpha^2 - k\,r^2\,.$$

The asymptotic expressions (23) and (32) are the simplest which discribe the focusing of a symmetry shock near focus, and they contain two characteristic parameters k and M_0. With

$$X = \sqrt{k}\,x \qquad\qquad Z = \sqrt{k}\,z$$
$$Y = \sqrt{k}\,y \quad \text{or} \quad R = \sqrt{k}\,r \tag{33}$$
$$T = \sqrt{k}\,\alpha \qquad\qquad T = \sqrt{k}\,\alpha$$

the expressions are written in "focusing scale": for the plane case

$$X = M_0 T - \tfrac{1}{3} M_0^2 G_0 T^3 - T Y^2, \qquad (34)$$
$$M = M_0 - M_0^2 G_0 T^2 - Y^2,$$

and for the axially symmetric case

$$Z = M_0 T - \tfrac{2}{3} M_0^2 G_0 T^3 - T R^2, \qquad (35)$$
$$M = M_0 - 2 M_0^2 G_0 T^2 - R^2,$$

thus $1/\sqrt{k}$ shows the scale of focusing and M_0 the level of focusing. For ideal gases

$$p = p_0 \frac{2\gamma M^2 - \gamma + 1}{\gamma + 1},$$

where p is the pressure behind the shock, p_0 is the pressure in the undisturbed medium ahead of the shock; therefore the shock pressure field near focus is given for the plane case by

$$p = p_f - \frac{4\gamma p_0}{\gamma + 1} M_0 k y^2 - \frac{4\gamma p_0}{\gamma + 1} M_0^3 G_0 k a_0^2 t^2 \qquad (36)$$

and for the axially symmetric case by

$$p = p_f - \frac{4\gamma p_0}{\gamma + 1} M_0 k r^2 - \frac{8\gamma p_0}{\gamma + 1} M_0^3 G_0 k a_0^2 t^2, \qquad (37)$$

where p_f is the pressure at focus

$$p_f = p_0 \frac{2\gamma M_0^2 - \gamma + 1}{\gamma + 1}.$$

6. Instability of Converging Cylindrical Shocks

Using the same method, the shock dynamics equations in polar coordinates (ϱ, φ) may be written for the plane case as

$$\varrho^2 \left(\frac{\partial \varrho}{\partial \alpha}\right)^2 = \varrho^2 M^2 + M^2 \left(\frac{\partial \varrho}{\partial \varphi}\right)^2,$$

$$\varrho^2 \left(\frac{\partial \varrho}{\partial \alpha}\right)^3 \frac{\partial M}{\partial \alpha} = M^2 \frac{\partial \varrho}{\partial \varphi} \frac{\partial M}{\partial \varphi} \left(\frac{\partial \varrho}{\partial \alpha}\right)^2 + M^5 G \frac{\partial^2 \varrho}{\partial \varphi^2} - 2 \varrho M^3 G \left(\frac{\partial \varrho}{\partial \alpha}\right)^2 + \varrho M^5 G, \qquad (38)$$

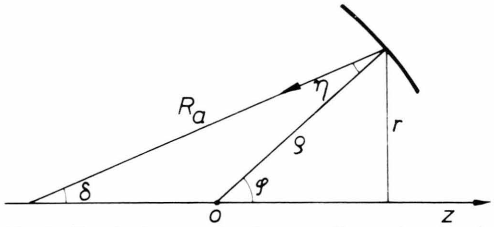


Fig. 2. Shock element for the two-dimensional axially symmetric case in polar coordinates.

here

$$\frac{1}{R_c} = - \frac{M^3 \left[\varrho^2 + 2\left(\frac{\partial \varrho}{\partial \varphi}\right)^2 - \varrho \frac{\partial^2 \varrho}{\partial \varphi^2}\right]}{\varrho^3 \left(\frac{\partial \varrho}{\partial \alpha}\right)^3} \qquad (39)$$

is substituted into (12).

For the axially symmetric case one obtaines (see Figure 2)

$$\frac{1}{R_a} = \frac{\sin \delta}{r} = \frac{\sin(\varphi - \eta)}{\varrho \sin \varphi} = \frac{1 - \operatorname{ctg} \varphi \operatorname{tg} \eta}{\varrho \sqrt{1 + \operatorname{tg}^2 \eta}}$$
$$= - M \left(1 - \frac{\operatorname{ctg} \varphi}{\varrho} \frac{\partial \varrho}{\partial \varphi}\right) \Big/ \varrho \frac{\partial \varrho}{\partial \alpha}. \qquad (40)$$

Substituting (39) and (40) into (24), the shock dynamics equations for the axially symmetric case in polar coordinates are given as

$$\varrho^2 \left(\frac{\partial \varrho}{\partial \alpha}\right)^2 = \varrho^2 M^2 + M^2 \left(\frac{\partial \varrho}{\partial \varphi}\right)^2,$$

$$\varrho^2 \left(\frac{\partial \varrho}{\partial \alpha}\right)^3 \frac{\partial M}{\partial \alpha} = M^2 \frac{\partial \varrho}{\partial \varphi} \frac{\partial M}{\partial \varphi} \left(\frac{\partial \varrho}{\partial \alpha}\right)^2 + M^5 G \frac{\partial^2 \varrho}{\partial \varphi^2} - 3 \varrho M^3 G \left(\frac{\partial \varrho}{\partial \alpha}\right)^2 + \varrho M^5 G + M^3 G \operatorname{ctg} \varphi \frac{\partial \varrho}{\partial \varphi} \left(\frac{\partial \varrho}{\partial \alpha}\right)^2. \qquad (41)$$

For converging circular cylindrical shocks, $\partial \varrho / \partial \varphi = 0$, (38) become

$$\frac{d\varrho}{d\alpha} = - M, \quad \frac{dM}{d\alpha} = \frac{M^2 G}{\varrho}, \qquad (42)$$

and the linear small perturbation equations are given by

$$M^* = - \partial \varrho^* / \partial \alpha,$$

$$\frac{\partial^2 \varrho^*}{\partial \alpha^2} - \left(2 + \frac{M}{G} \frac{dG}{dM}\right) \frac{MG}{\varrho} \frac{\partial \varrho^*}{\partial \alpha} - \frac{M^2 G}{\varrho^2} \left(\varrho^* + \frac{\partial^2 \varrho^*}{\partial \varphi^2}\right) = 0. \qquad (43)$$

A converging cylindrical shock will ultimately become strong, so that for the stability investigation it is sufficient to consider the case of strong shocks, $M \to \infty$, and from the Chester-Chisnell relation

$$\frac{1}{G(M)} = \frac{2M^2}{K(M)(M^2-1)} \to \frac{\gamma+2}{\gamma} + \sqrt{\frac{2\gamma}{\gamma-1}} := n$$

($n \doteq 5.0743$ for $\gamma = 1.4$).

Taking the time, where the shock arrives at the centre of the circle, as zero, we have solutions in Guderley-form [5] from (42):

$$\varrho = c\,(-\alpha)^{n/(n+1)}\,, \qquad M = \frac{c\,n}{n+1}(-\alpha)^{-1/(n+1)}\,, \tag{44}$$

where c is an arbitrary constant. To obtain the small perturbation equation for strong shocks, G in (43) cannot directly set equal $1/n$, however $\frac{M}{G}\frac{\mathrm{d}G}{\mathrm{d}M}$ has to be determined by infinitesimal analysis (see also Fig. 3):

$$\frac{\mathrm{d}G}{\mathrm{d}M} = O\left(\frac{1}{M^3}\right), \quad \frac{M\,\mathrm{d}G}{G\,\mathrm{d}M} \to 0\,.$$

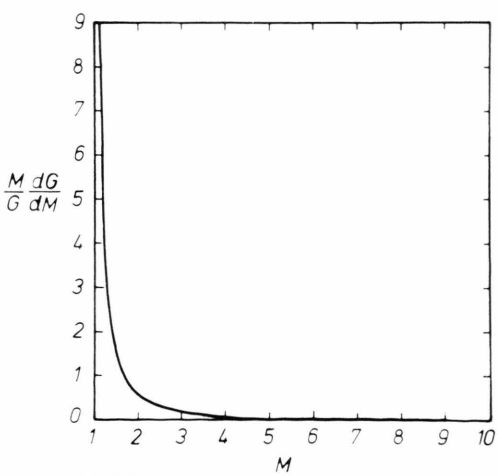


Fig. 3. $\frac{M}{G}\frac{\mathrm{d}G}{\mathrm{d}M}$ as a function of M.

The linear small perturbation equation is given by

$$\frac{\partial^2\varrho^*}{\partial\alpha^2} + \frac{2}{(n+1)\,\alpha}\frac{\partial\varrho^*}{\partial\alpha} - \frac{n}{(n+1)^2\,\alpha^2}\left(\varrho^* + \frac{\partial^2\varrho^*}{\partial\varphi^2}\right) = 0\,. \tag{45}$$

With solutions of the form

$$\varrho^* = (-\alpha)^l/e^{im\varphi} \tag{46}$$

substituted into (45) one gets

$$l\,(l-1) + \frac{2l}{n+1} - \frac{n}{(n+1)^2}(1-m^2) = 0 \tag{47}$$

or

$$l = \frac{(n-1) \pm \sqrt{(n+1)^2 - 4n\,m^2}}{2\,(n+1)}\,, \tag{48}$$

which is identical with the result obtained by Gardner, Book & Bernstein [6] ($n \leftrightarrow \beta$ in [6], $l \leftrightarrow \beta$ in [6]).

For $m = 0$ $\quad l = \begin{cases} \dfrac{n}{n+1} \\ -\dfrac{1}{n+1} \end{cases}$,

the shocks converge still symmetrically and keep in a circle.

For $m = 1$ $\quad l = \begin{cases} \dfrac{n-1}{n+1} \\ 0 \end{cases}$,

under the approximation of linear small perturbations only the center of the shock is disturbed. The symmetry of the shock is disturbed just only for $m \geqq 2$, with l becoming complex:

$$\mathrm{Re}\,(l) = \frac{1}{2}\,\frac{n-1}{n+1}\,, \tag{49}$$

$$\mathrm{Im}\,(l) = \frac{\sqrt{4n\,m^2 - (n+1)^2}}{2\,(n+1)} := \omega\,. \tag{50}$$

The perturbation oscillates with the frequency ω (in $\ln(-\alpha)$) which depends on the mode number m, and the amplitude changes independently of m. Because

$$\mathrm{Re}\,(l) \leqq n/(n+1)$$

the amplitude of perturbation increases relatively and the converging shock is geometrically unstable near the center.

The characteristic angle, along which the small perturbation propagates on the shock, is given by $\tan^{-1}\sqrt{G}$; with the strong shock approximation this is constant, i.e. $\tan^{-1}\sqrt{1/n}$; thus the loci of perturbation for converging circular cylindrical shocks are logarithmic spirals (see Fig. 4):

$$\varrho = c\, e^{\pm\sqrt{n}\,\varphi}, \tag{51}$$

where c is an arbitrary constant. With (44) one obtains the interval of time in which a small perturbation propagates through an angle $\Delta\varphi$:

$$\Delta \ln(-\alpha) = \frac{n+1}{\sqrt{n}}\,\Delta\varphi \tag{52}$$

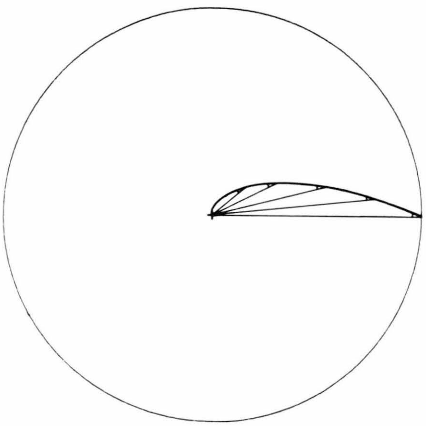


Fig. 4. Locus of perturbation on a converging cylindrical shock.

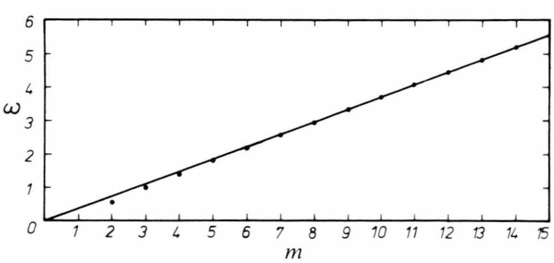


Fig. 5. *Dots:* Frequency ω of small perturbation as a function of mode number m (cf. (50)). – *Line:* Reciprocal time of perturbation propagation along a circular shock for an angle equivalent to one mode sector (cf. (53)).

or in $\omega - m$ form $\left(\Delta \ln(-\alpha) = \frac{2\pi}{\omega},\ \Delta\varphi = \frac{2\pi}{m}\right)$

$$\omega = \frac{\sqrt{n}}{n+1}\, m\,. \tag{53}$$

The comparison between (50) and (53) is plotted in Figure 5.

7. Numerical Experiment

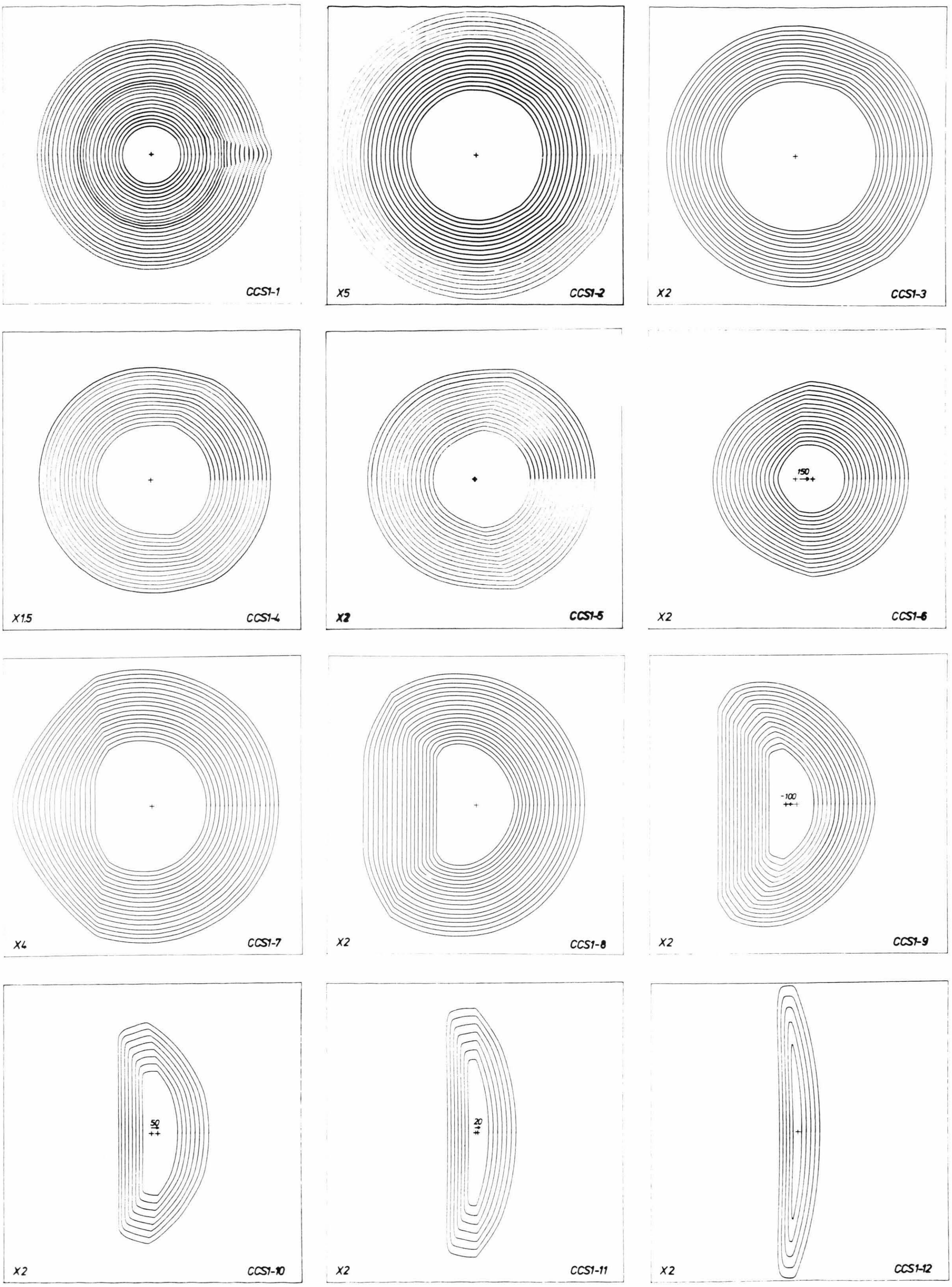
When a cylindrical shock converges, small perturbations on it increase relatively. The development of the perturbations is studied by a numerical method. According to the CCW approach shock-elements move in the direction of the shock normal and the change of M depends on the change of A (the integral relation $A \sim M$) or on the radius of curvature R_c (the differential relation (12) or (13), or (24) for the axially symmetric case).

Davies and Guy [7], Fong and Ahlborn [8], and Gardner, Book, and Bernstein [6] have made numerical calculations using the CCW approach. Here we describe the shock-shape near a point on the shock by a 2-power expression (with the two nearest shock points on both sides) to obtain the direction of the normal and the radius of curvature at that point. Of course, the Courant condition has to be satisfied and the grid points should be redistributed.

Figure 6 shows a result of the numerical experiment. At the beginning a shock of Mach number $M=8$ is assumed on a radial distribution in the following form:

$$\varrho = \begin{cases} \varrho_0 = \text{constant} \\ \qquad \text{for } -\pi \leqq \varphi \leqq -\frac{\pi}{16} \text{ and } \frac{\pi}{16} \leqq \varphi \leqq \pi, \\ \varrho_0\,[1 + 0.025\,(1+\cos 16\,\varphi)] \\ \qquad \text{for } -\frac{\pi}{16} \leqq \varphi \leqq \frac{\pi}{16}\,. \end{cases}$$

The numerals in the left-lower corner are magnifying powers. In order to continue the calculation the polar origin is sometimes shifted to the middle. Figures 7–9 are other examples. Figure 10 shows the maximum of relative disturbances. Because of the forming of shock-shocks, cumulation of numeri-


CCS1-1
X5
CCS1-2
X2
CCS1-3
X1.5
CCS1-4
X2
CCS1-5
150
X2
CCS1-6
X4
CCS1-7
X2
CCS1-8
-100
X2
CCS1-9
50
X2
CCS1-10
20
X2
CCS1-11
X2
CCS1-12

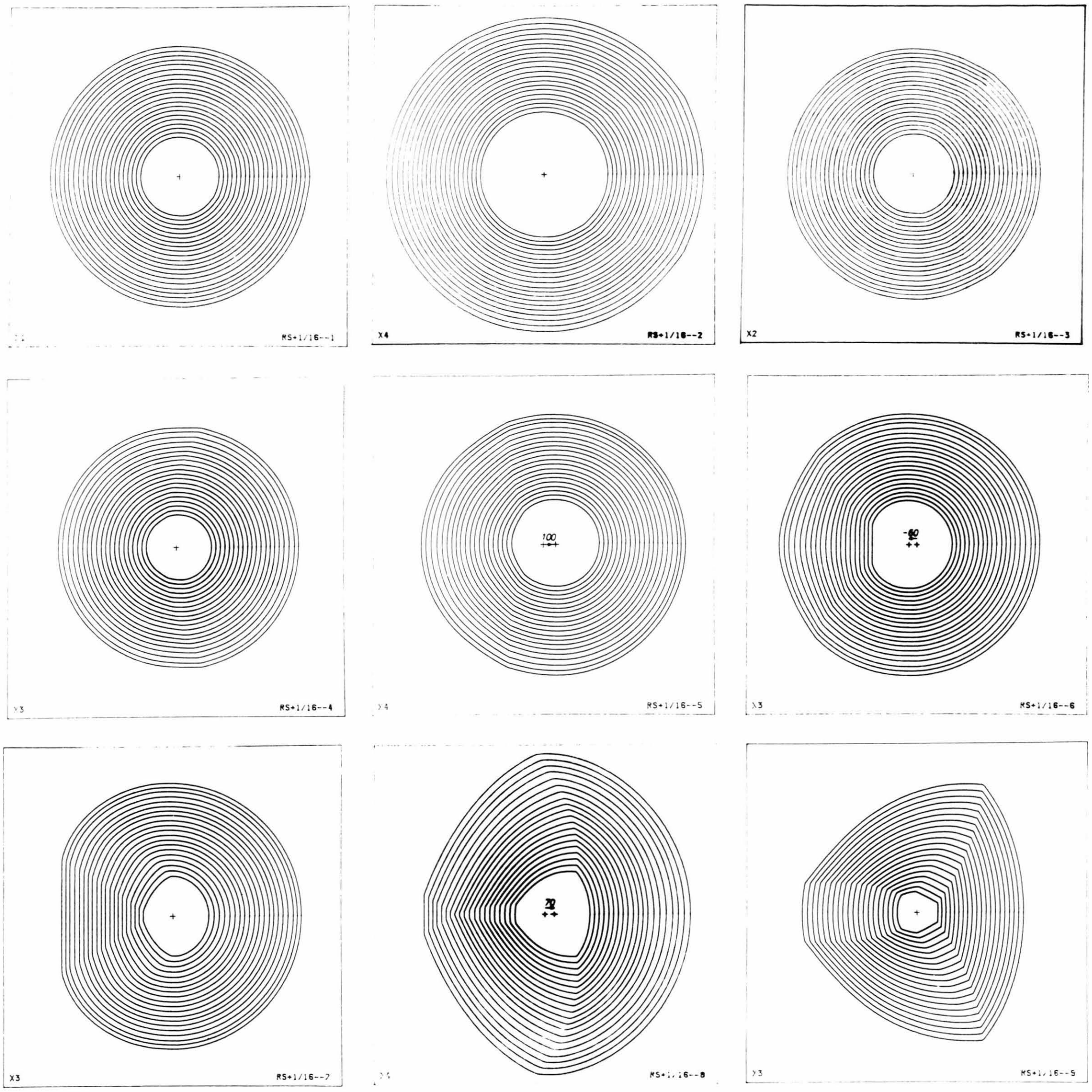


Fig. 7. Beginning with $M = 10$:

$$\varrho = \begin{cases} \varrho_0 = \text{const}\,, \\ \varrho_0\,[1 + 0.005\,(1 + \cos 16\,\varphi)] \quad \text{for} \quad -\dfrac{\pi}{16} \leqq \varphi \leqq \dfrac{\pi}{16}\,. \end{cases}$$

Fig. 6. Converging cylindrical shock with perturbation. Beginning with $M = 8$:

$$\varrho = \begin{cases} \varrho_0 = \text{const}\,, \\ \varrho_0\,[1 + 0.025\,(1 + \cos 16\,\varphi)] \quad \text{for} \quad -\dfrac{\pi}{16} \leqq \varphi \leqq \dfrac{\pi}{16}\,. \end{cases}$$

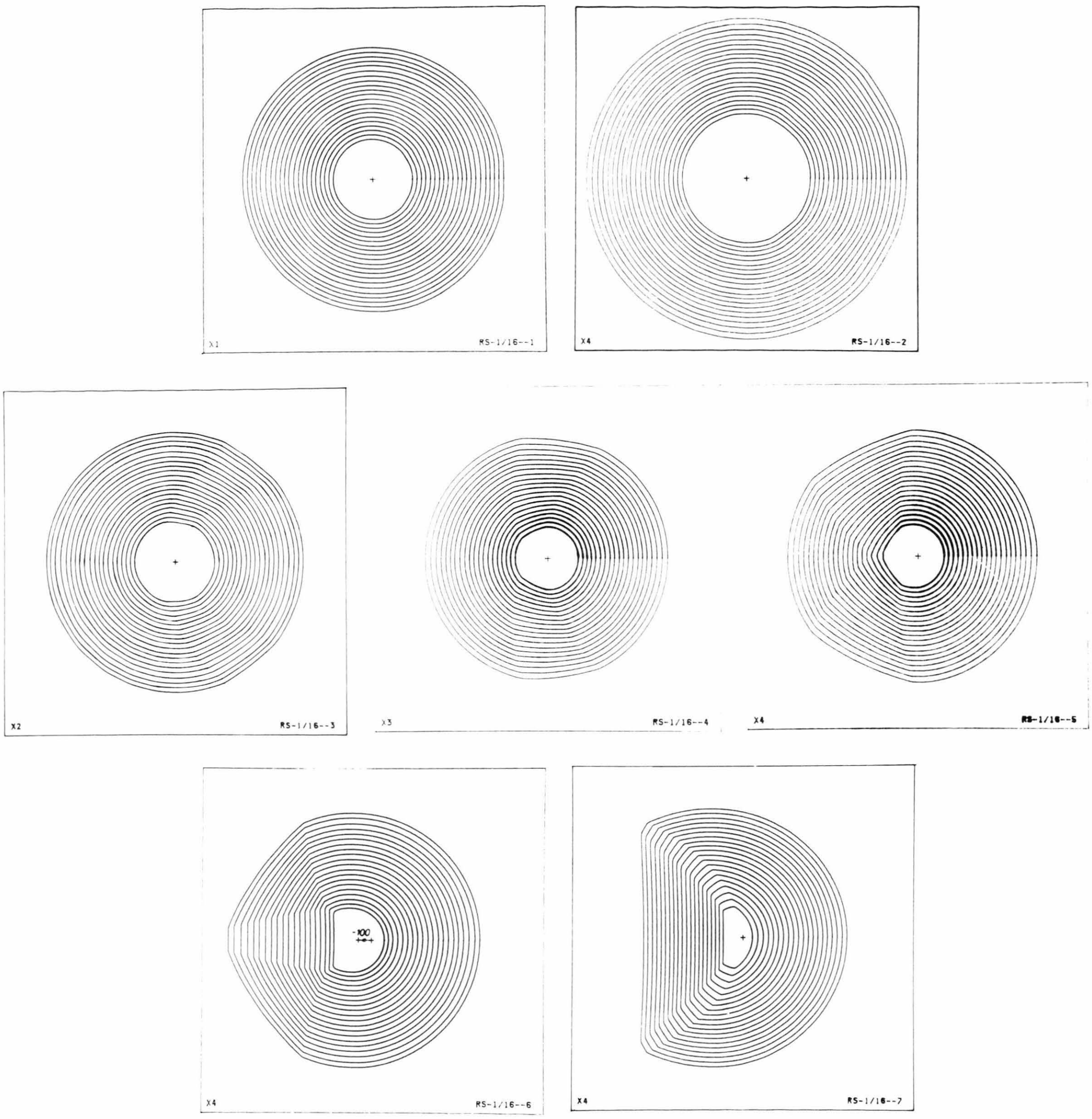
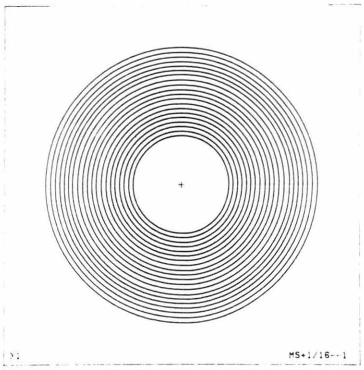


Fig. 8. Beginning with $M = 10$:

$$\varrho = \begin{cases} \varrho_0 = \text{const}\,, \\ \varrho_0\,[1 - 0.005\,(1 + \cos 16\,\varphi)] \quad \text{for} \quad -\dfrac{\pi}{16} \leqq \varphi \leqq \dfrac{\pi}{16}\,. \end{cases}$$



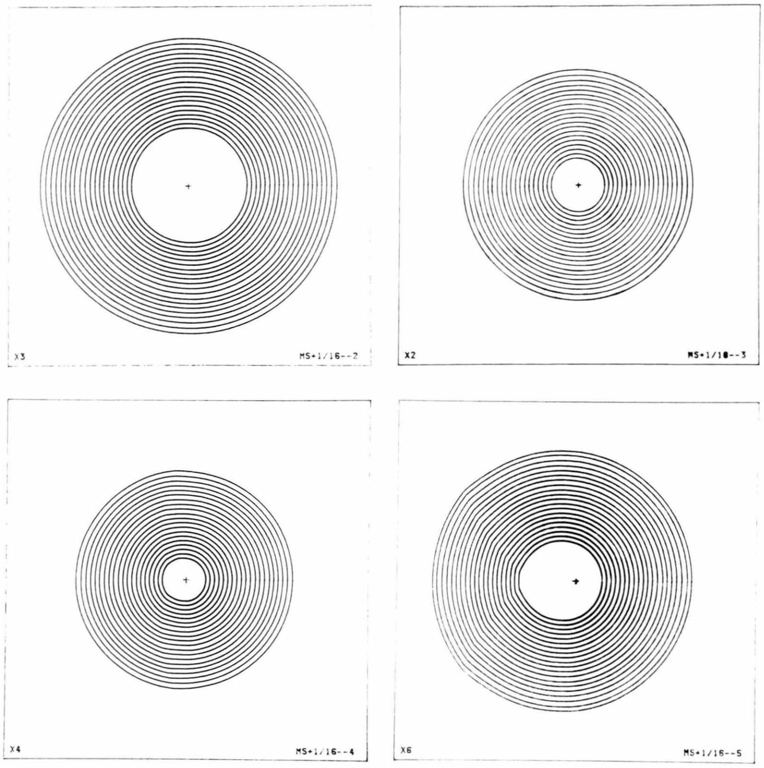


Fig. 9. Beginning with $\varrho = \text{const}$:

$$M = \begin{cases} 10\,, \\ 10\,[1 + 0.005\,(1 + \cos 16\,\varphi)] \end{cases}$$

$$\text{for} \quad -\frac{\pi}{16} \leqq \varphi \leqq \frac{\pi}{16}\,.$$

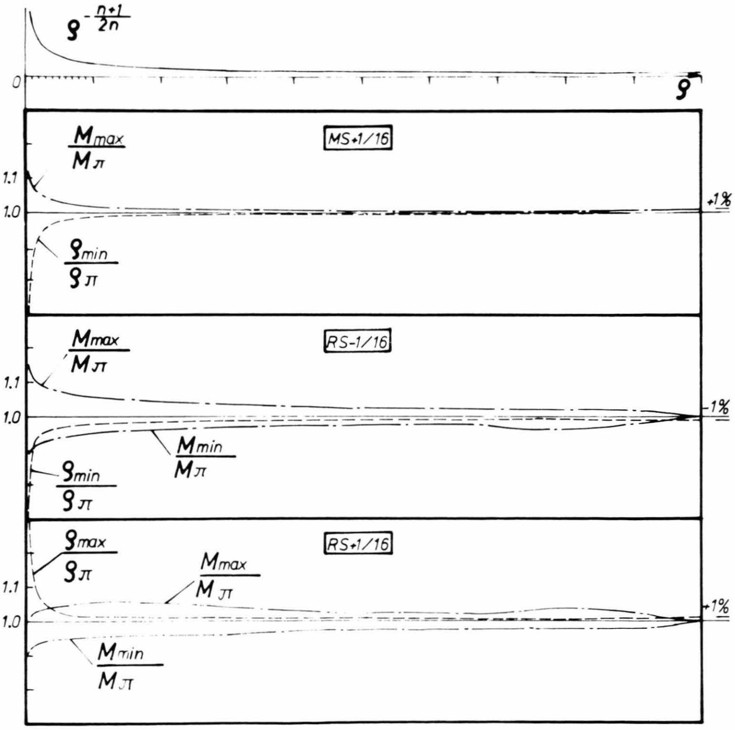


Fig. 10. Development of the maximum of relative disturbance. On the top it is from linear small perturbation analysis.

cal errors and rapid reduction of the shock extension, one cannot expect to obtain exact pictures and quantitative results, but only would like to get some qualitative cognition.

A converging cylindrical shock is unstable near the center. But the development of perturbations is not serious; it manifests itself remarkably only in the small time-space range near the axis, if according to the linear small preturbation theory the amplitude of relative perturbations changes as $\varrho^{-(n+1)/(2n)}$ ($\varrho^{-0.6}$ for $\gamma = 1.4$).

To assure a relative precision ε, the perturbation ought to be restricted to $\varepsilon^{(3n+1)/(2n)}$ ($\varepsilon^{1.6}$ for $\gamma = 1.4$). Though a perturbation can be propagated in cycles along a circular shock, random perturbations cannot form an intrinsically stable asymptotic picture, since the shock collapses and they do not have sufficient time.

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